

# Applied population analysis, week 2

WLF 504, Fall 2019

# Given our data, what is the most likely estimate of survival?

- This is, in a nutshell, maximum likelihood estimation (MLE) of a parameter (survival).
- “Using mathematics, maximum likelihood estimation is a method to estimate the most likely value of a parameter, given a sample of data” - *Powell and Gale*

# Two steps to using an MLE

1. Write an MLE statement (example to follow), describing the likelihood function.
2. Maximize the likelihood function, using calculus (but the computer mostly does this for us, so it's ok. Example to follow).

# An example

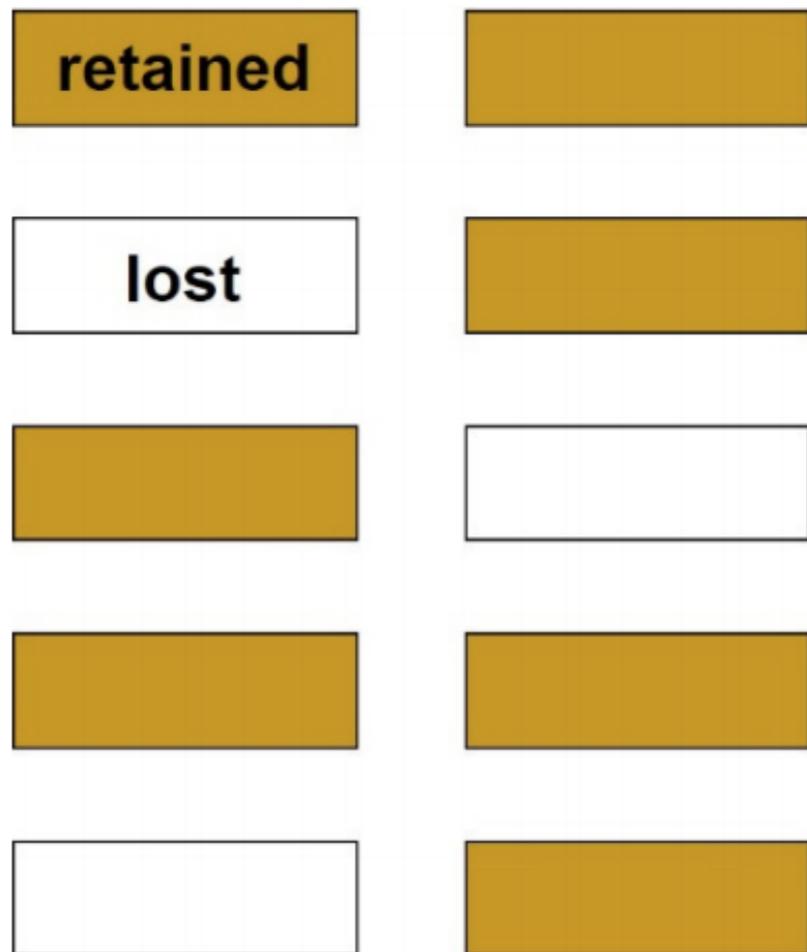
- Many population parameters can be thought of as the outcome of a series of binomial trials (0, 1 outcome).
- We can think of the outcome of multiple binomial trials as  $y$  successes out of  $n$  trials, where:

$$0 \leq y \leq n.$$

- We will call the probability of success  $p$ , and the probability of failure  $q$ , and of course  $p = 1 - q$

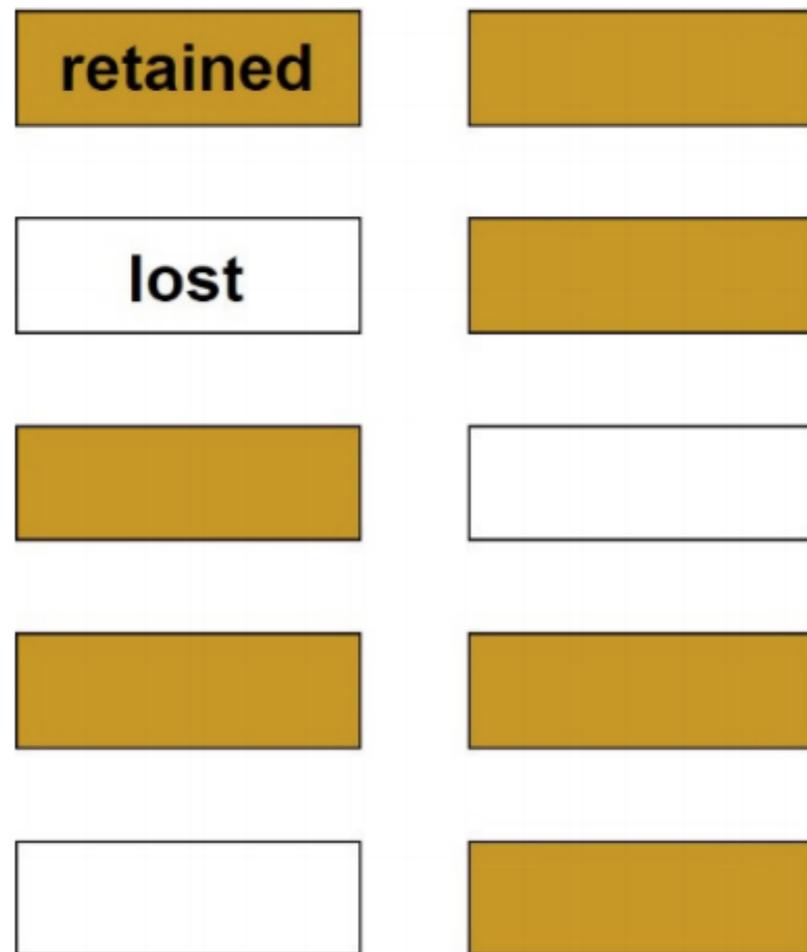
# An example

- Let's say we want to know if a cool new tag for marking birds will stay put over time. We mark 10 birds, and a week later, 7 still have tags



# An example

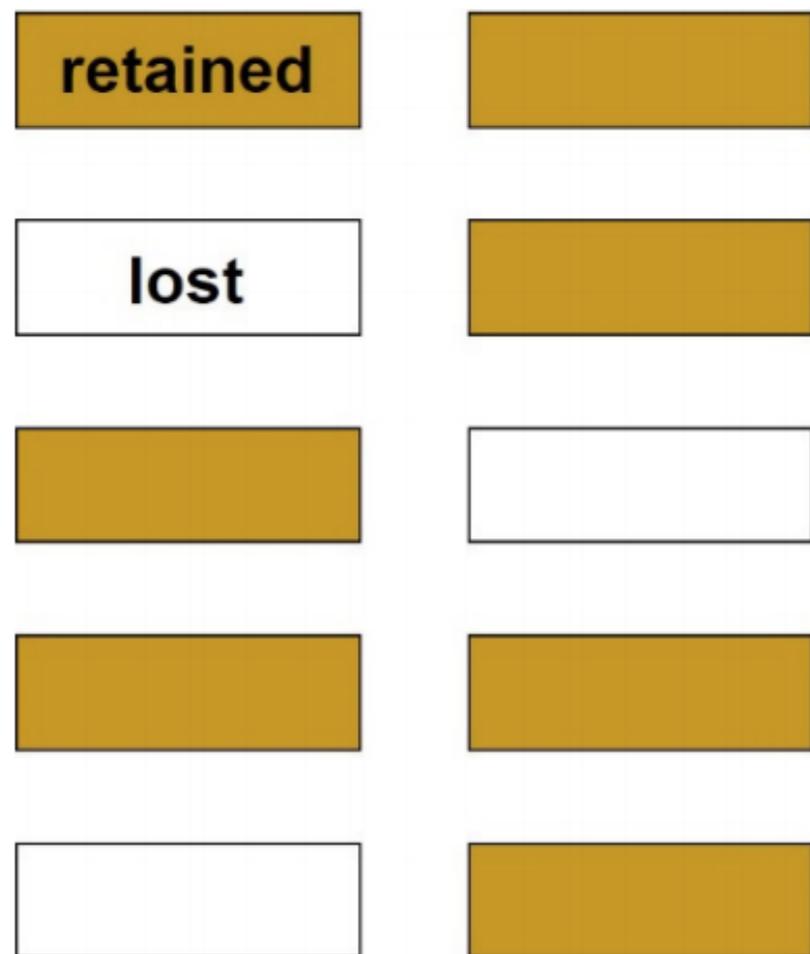
- Let's say we want to know if a cool new tag for marking birds will stay put over time. We mark 10 birds, and a week later, 7 still have tags



What's the probability that the tags are "successful" and stay put?

# An example

- Let's say we want to know if a cool new tag for marking birds will stay put over time. We mark 10 birds, and a week later, 7 still have tages



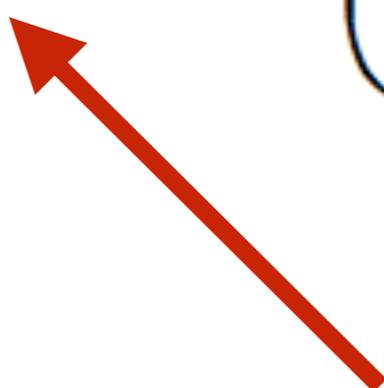
What's the probability that the tags are "successful" and stay put?

0.7, you say?! OK, yes.

But let's calculate this using maximum likelihood estimation (MLE)

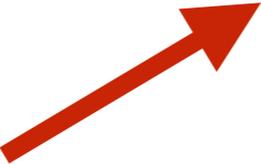
# An example

$$L(y | N, p) = \binom{N}{y} p^y (1-p)^{(N-y)}$$



“the likelihood of observing  $y$  successes, given  $N$  trials and the presence of a parameter,  $p$ , that we want to estimate equals...”

# An example

$$L(y | N, p) = \binom{N}{y} p^y (1-p)^{(N-y)}$$


We use the binomial coefficient to calculate how many ways (combinations) you could get what you are proposing (in our case: how many ways could 7 tags be retained on 10 birds?)

$$\binom{N}{y} = \frac{N!}{y!(N-y)!}$$

$$N! = N * (N-1) * (N-2) * \dots * 1$$

# An example

$$L(y | N, p) = \binom{N}{y} p^y (1-p)^{(N-y)}$$



- Probability of getting our exact result (7 successes and 3 failures) in any order (could be tags 1-3 that fail, or 3,6, and 9, or ...).

# An example

$$L(y | N, p) = \binom{N}{y} p^y (1-p)^{(N-y)}$$


- Need to raise  $p$  (prob of 1 success) to the number of success to get prob. of  $y$  successes, and  $q$  (which is  $1-p$ ) to the number of failures to get prob. of  $N-y$  failures.

# An example

$$L(y | N, p) = \binom{N}{y} p^y (1-p)^{(N-y)}$$



$$L(\theta | y = 7) = \binom{10}{7} p^7 (1-p)^{10-7}$$

- Now, we re-write this equation to ask, what is the likelihood of a value (theta) of our parameter, given the data that we got ( $y = 7$  successes)?

Next we simplify...

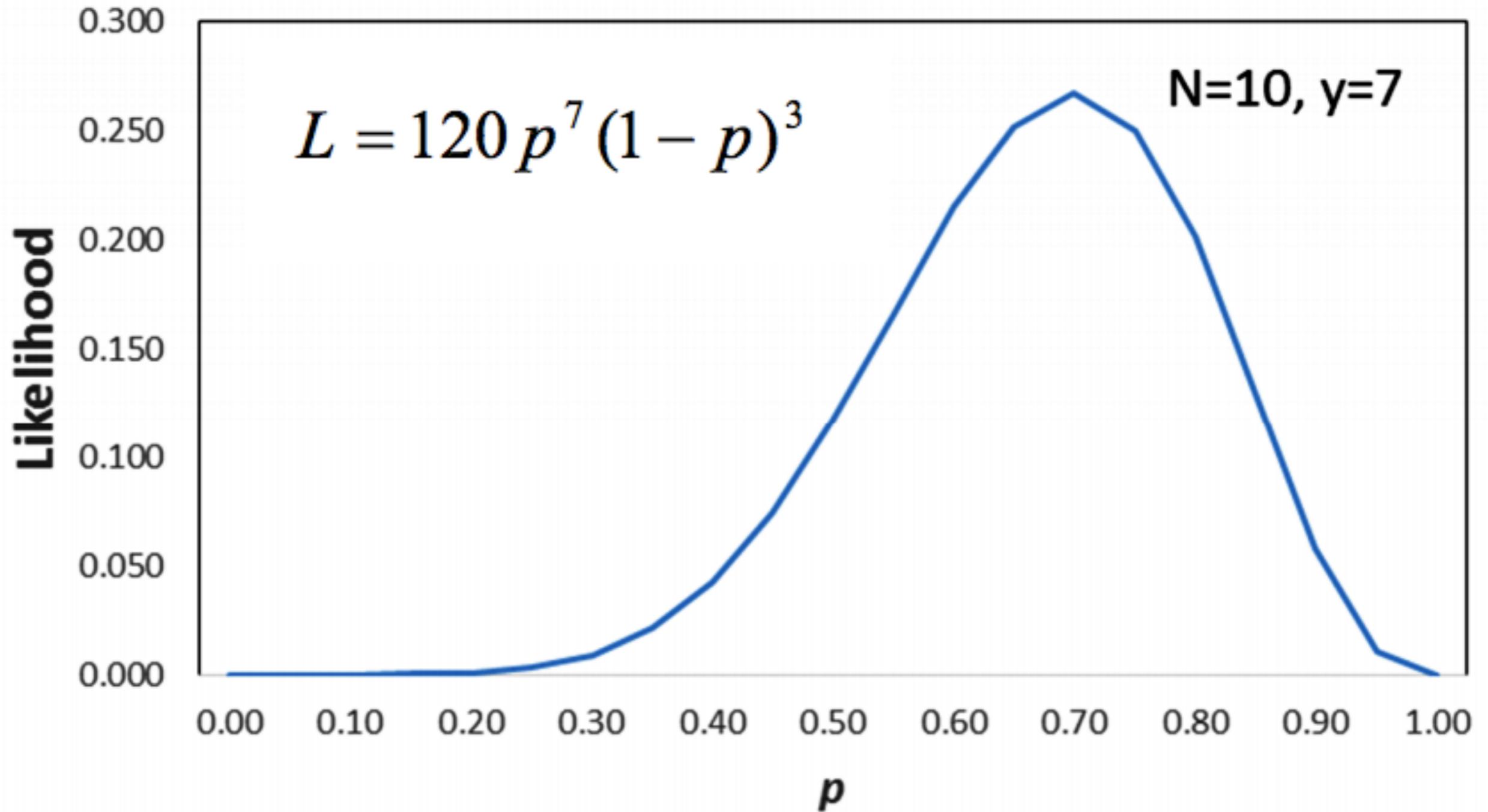
$$L(\theta | y = 7) = \binom{10}{7} p^7 (1-p)^{10-7}$$

$$L(\theta | y = 7) = \left[ \frac{10!}{7!(10-7)!} \right] p^7 (1-p)^3$$

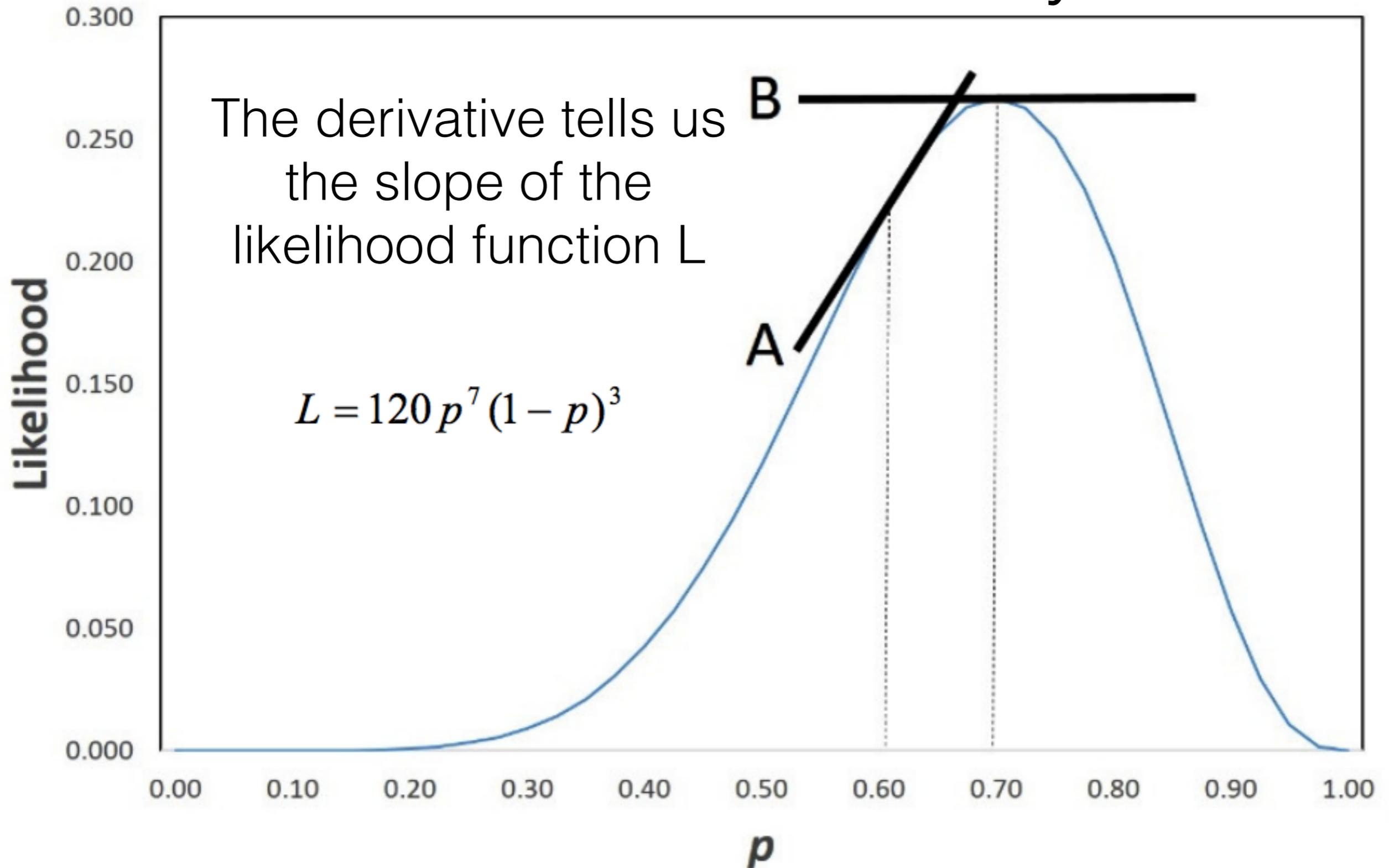
$$L(\theta | y = 7) = \left[ \frac{10 * 9 * 8}{3 * 2 * 1} \right] p^7 (1-p)^3$$


$$L(\theta | y = 7) = 120 p^7 (1-p)^3$$

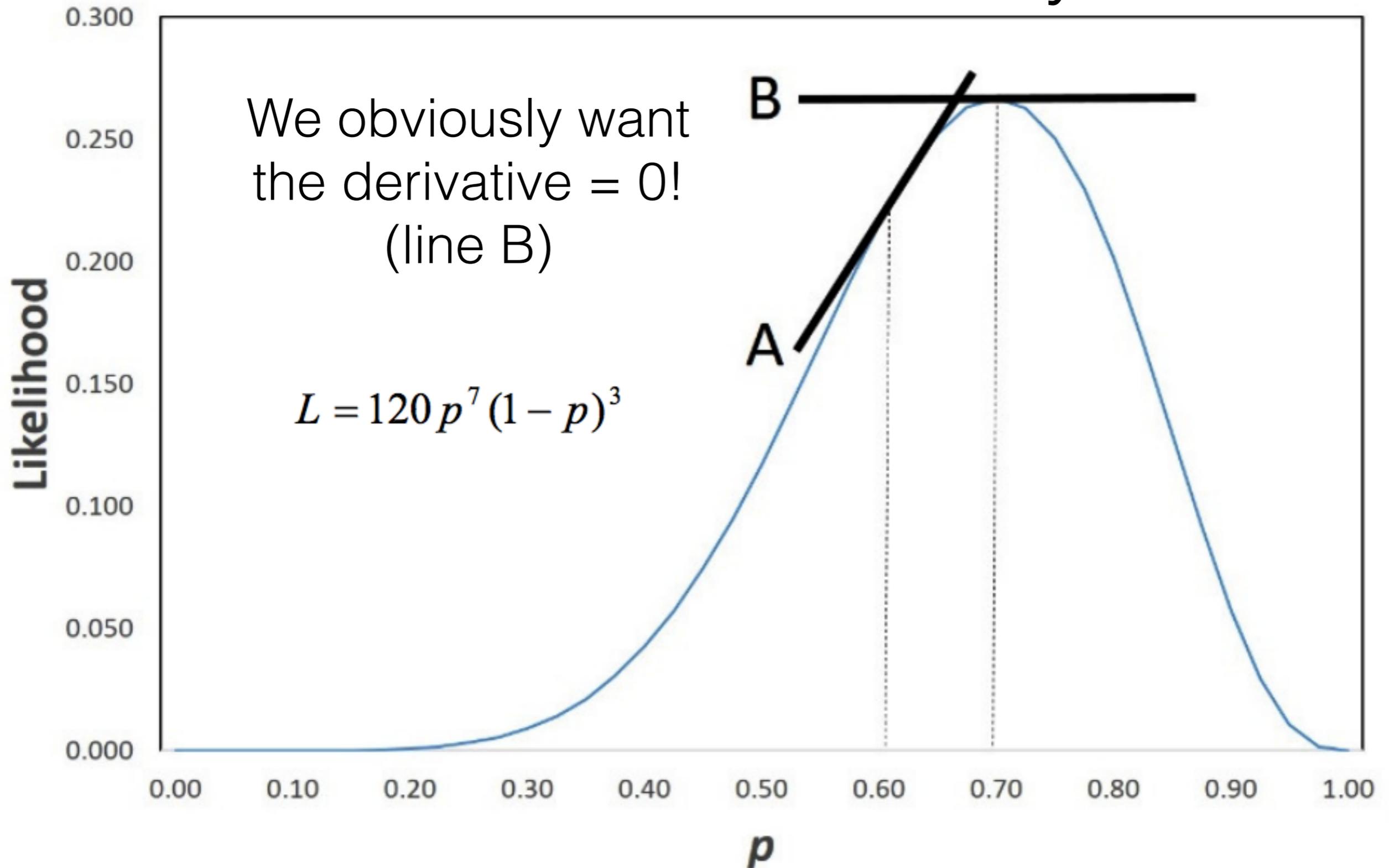
And then we “maximize” the likelihood numerically



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And then we “maximize” the likelihood numerically



# How to approximate the slope (derivative)?

Remember back in the day?

- Product rule  $(f \cdot g)' = f' \cdot g + f \cdot g'$
- Power rule  $f'(x) = rx^{r-1}$

$$(120p^7(1-p)^3)' = 0$$



$$840(1-p)^3 \cdot p^6 - 360(1-p)^2 \cdot p^7 = 0$$

# Simplify...

$$\frac{840}{360}(1-p) = p$$

$$2.333 - 2.333p = p$$

$$2.333 = 3.333p$$

$$\hat{p} = 0.7$$



Why do we talk about the  
“log likelihood” thing?

$$L(y | N, p) = \binom{N}{y} p^y (1-p)^{(N-y)}$$



$$\ln L(p | data) = y \ln p + (N - y) \ln(1 - p)$$

**Taking the natural log gets rid of  
exponents, simplifying the analytical  
process**

Ok, now we take the derivative, and set to 0 (the “max” of the original probability distribution)


$$\frac{\delta[\ln L(p | data)]}{\delta p} = \frac{y}{p} - \frac{(N - y)}{(1 - p)}$$

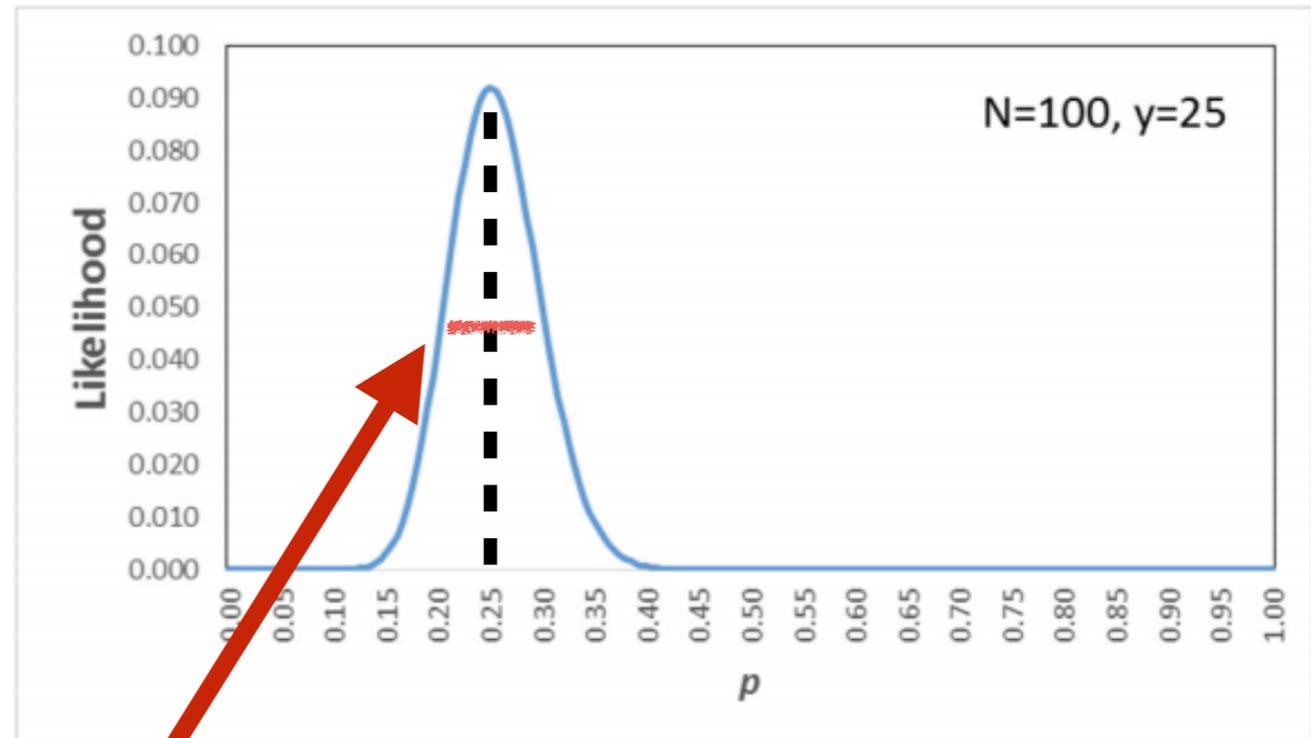
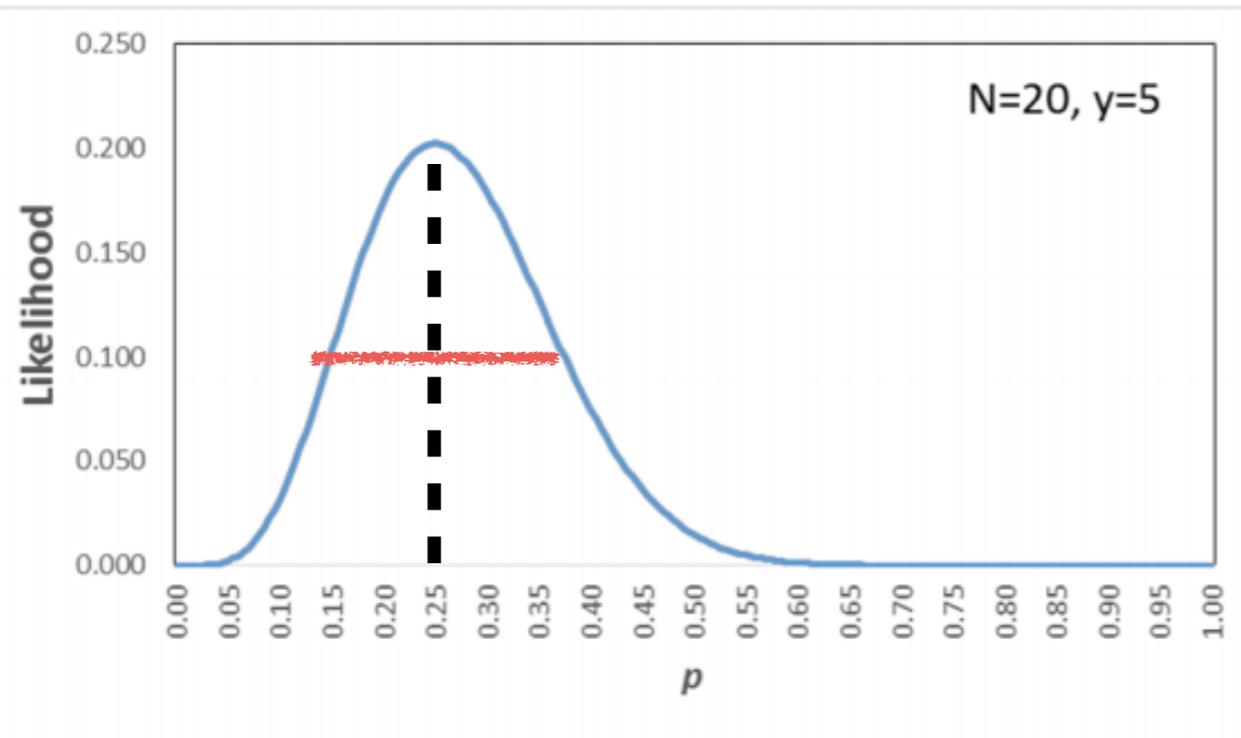
$$\frac{y}{p} - \frac{(N - y)}{(1 - p)} = 0$$

$$\hat{p} = \frac{y}{N}$$

**Wow, this looks a lot like the known-fates Kaplan-Meier survival estimator!**

$$\hat{S}_t = \frac{l_t - d_t}{l_t}$$

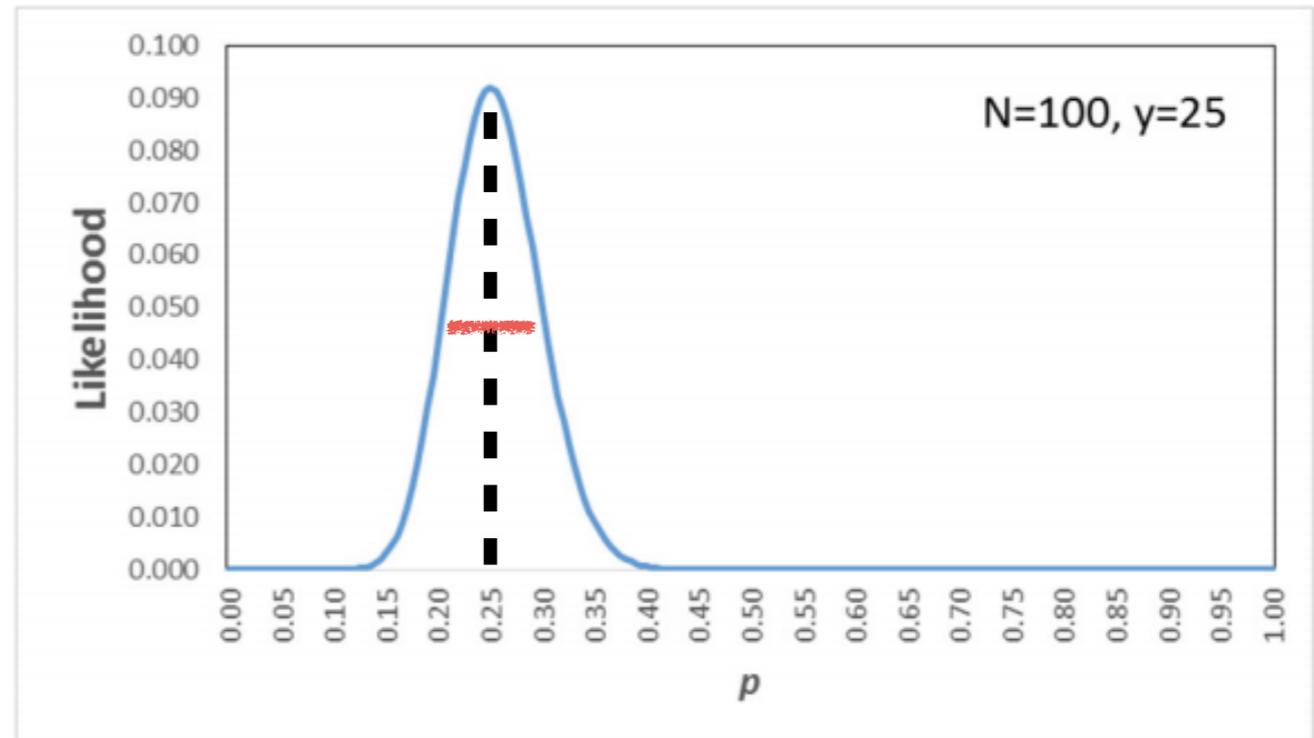
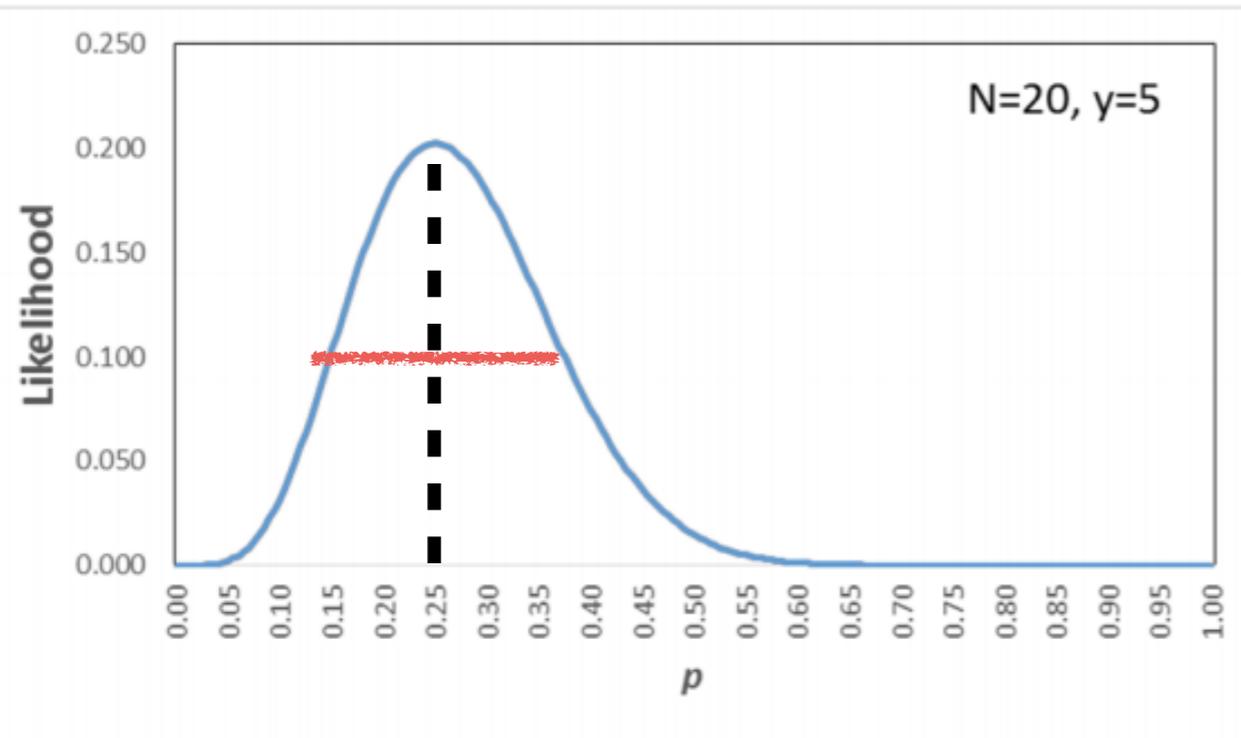
# Let's talk about variance real fast...



$$\hat{p} = 0.25 \text{ for both...}$$

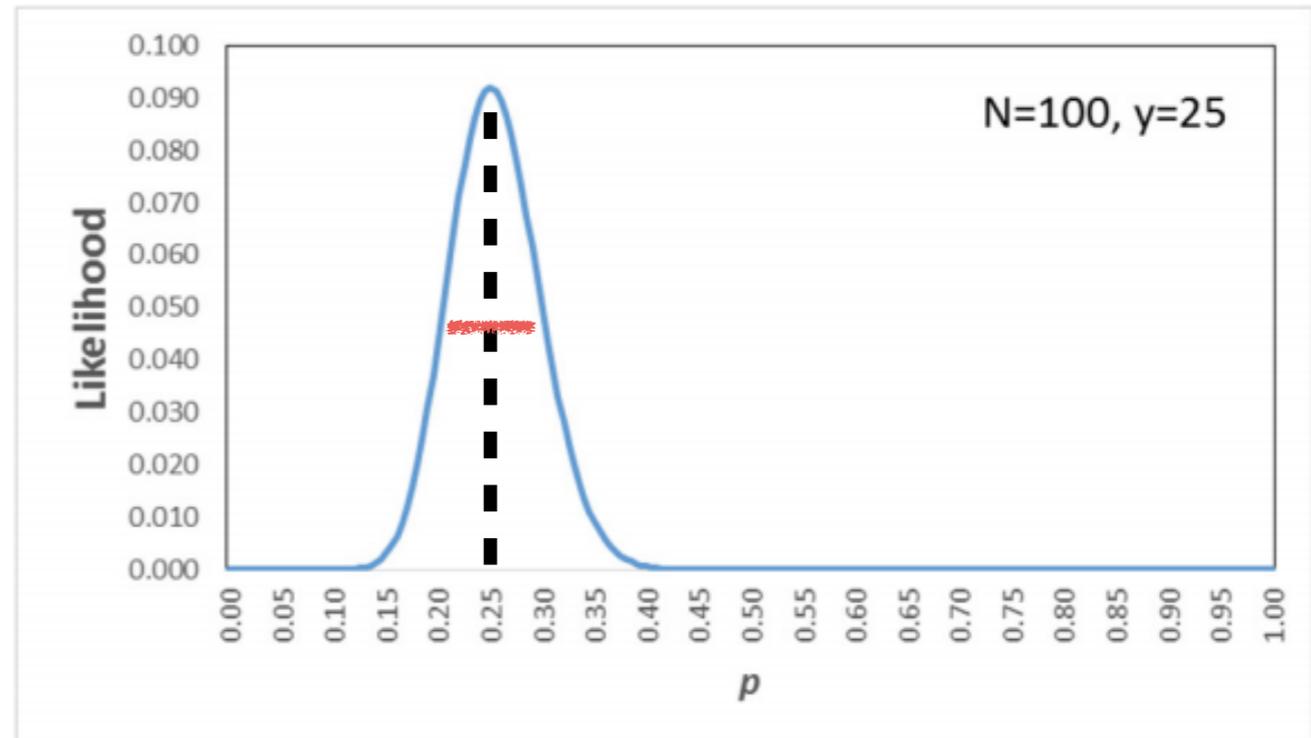
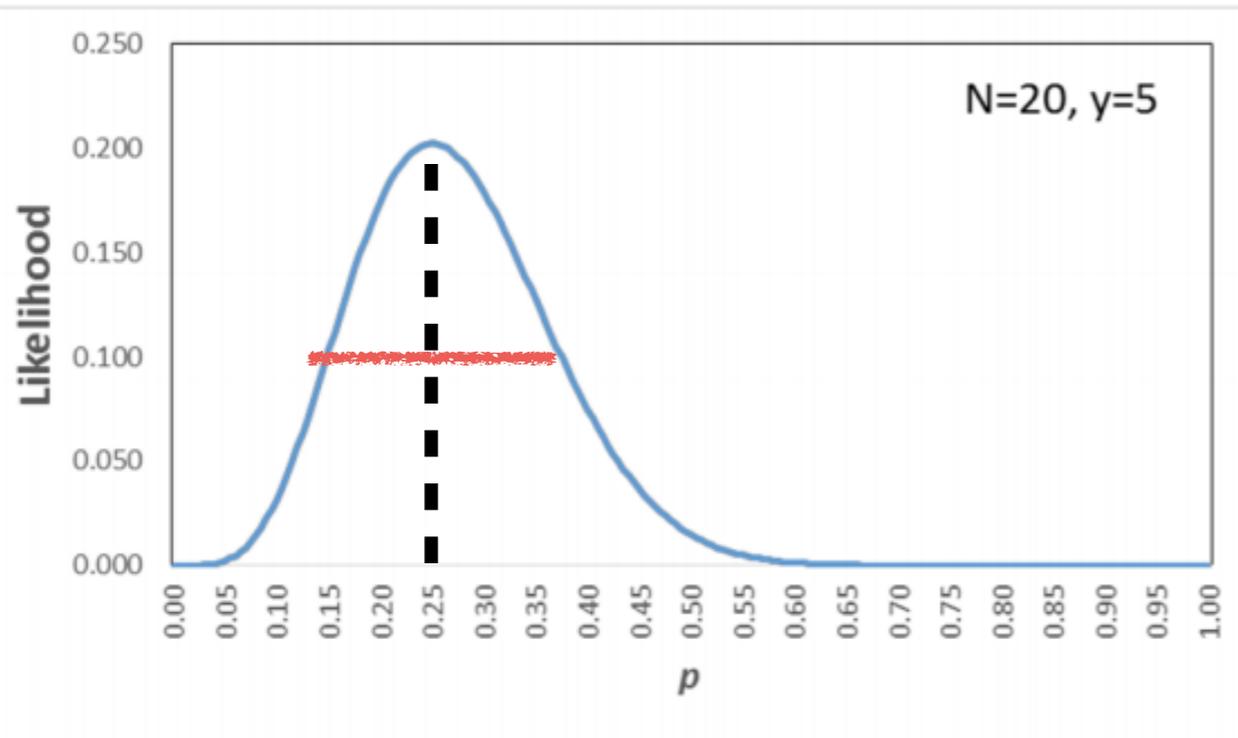
But a much **narrower range** of values of  $p$  are likely when  $N$  is larger- steeper **curve!**

# Let's talk about variance real fast...



- Second derivatives (derivative of the derivative”) tell us about **curvature** of the line of a function.

# Let's talk about variance real fast...



- The negative inverse of the second derivative = MLE estimate of the variance

$$\hat{p} = \frac{y}{N}$$

$$\text{var}(\hat{p}) = \frac{p(1-p)}{N}$$

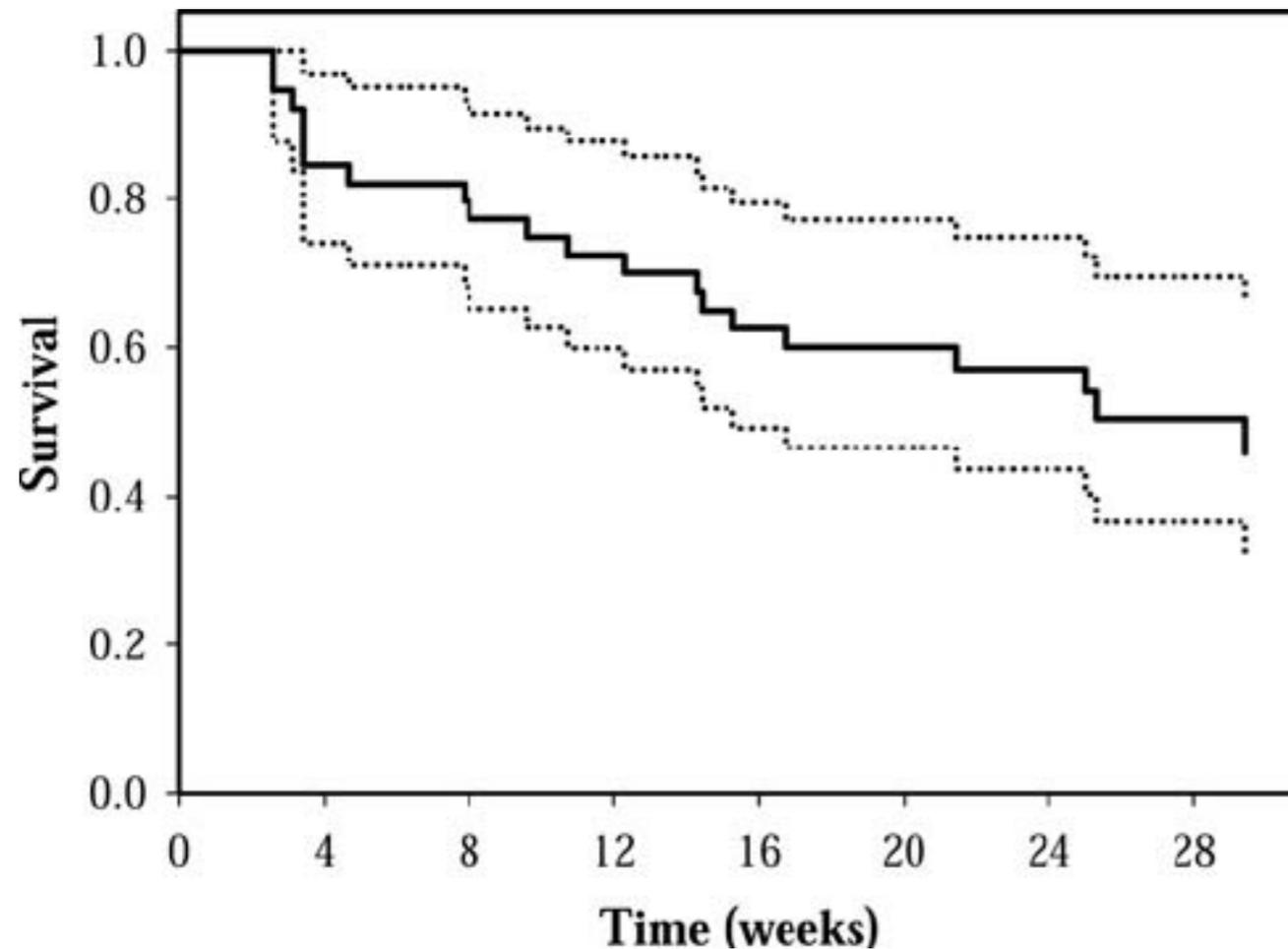
Quick reminder: AIC-based model selection derives from MLE

$$AIC = -2 \ln(L) + 2 * \text{number of parameters}$$

**First term: adequacy of model fit**

**Second term: penalty for model complexity**

# How can we estimate survival?



Survival to time  $t$  is a series of shorter-duration survival outcomes:

$$S(t) = S_1 * S_2 * \dots * S_{t-1}$$

But how do we estimate the survival for each interval?

# Capture-recapture analysis

- Anytime we mark an animal (can be individually recognizable or not)
- Various techniques for closed vs. open populations (which mostly means, in practice, how short vs. long in duration is your sampling)
- Also various techniques based on whether detection of marked animals in subsequent sampling sessions is  $=1$  (known fate) or  $< 1$ .

# A lot of different Q's can be asked w/CMR models

- Reverse time models (Pradel, Link-Barker JS) — we want to estimate probability of animals **entering** the population (**recruitment**)
- Band/Tag-recovery models- can estimate the probability of permanently leaving the study area (**emigration**)

# Latest-and-greatest

- Improve survival estimates by incorporating information from other data sources (e.g., from count data, in IPMs) using Bayesian tools
- Including evolutionary effects, such as effects of genotypes on survival and fitness (including in population viability analysis)
- Extension of hazard/failure-time models to data sets where only some fates are known using Bayesian tools

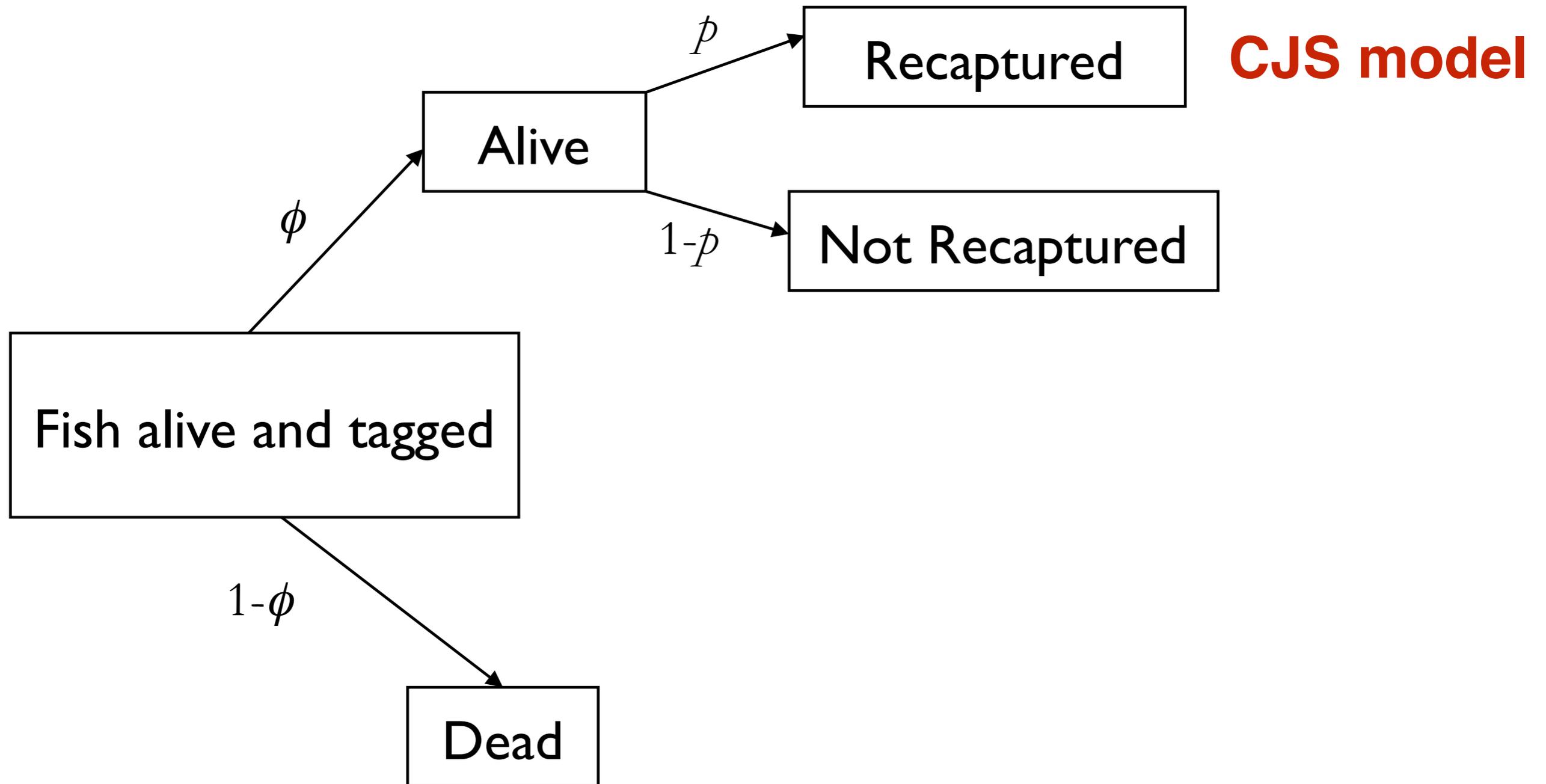
# Today: CJS model

- Can't tease apart death from permanent emigration (generally underestimates true survival)
- Unmarked animals not part of likelihood
- No estimation of abundance or recruitment

# Today: CJS model

- Sampling conducted over a small area on at least 3 occasions (e.g., years)
- Release  $R_i$  animals each occasion  $i = 1 \dots, k$
- Recaptures = could be handling or re-sighting (telemetry)
- Parameters:
  - Capture probability,  $p_i$ : prob. that marked animal is captured in period  $i$  (so,  $p_1$  not modelled, no info)
  - Apparent survival,  $\phi_i$  ( $\phi$ ): prob. that an animal alive in time  $i$  survives until  $i + 1$  and does not permanently emigrate

# Probability diagrams



# Probability diagrams

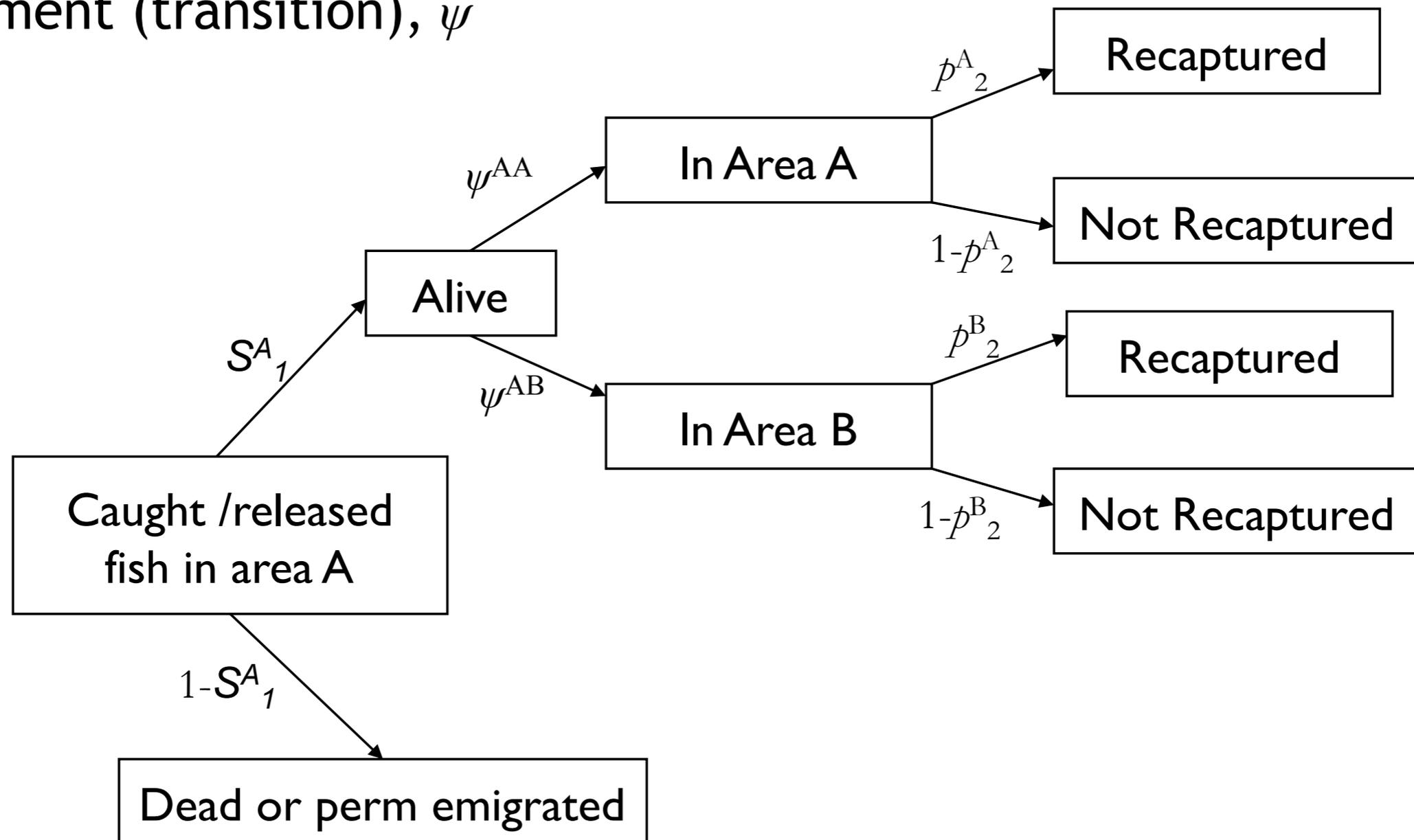
**Parameters** (area, time indexed)

Capture probability,  $p$  (area, time)

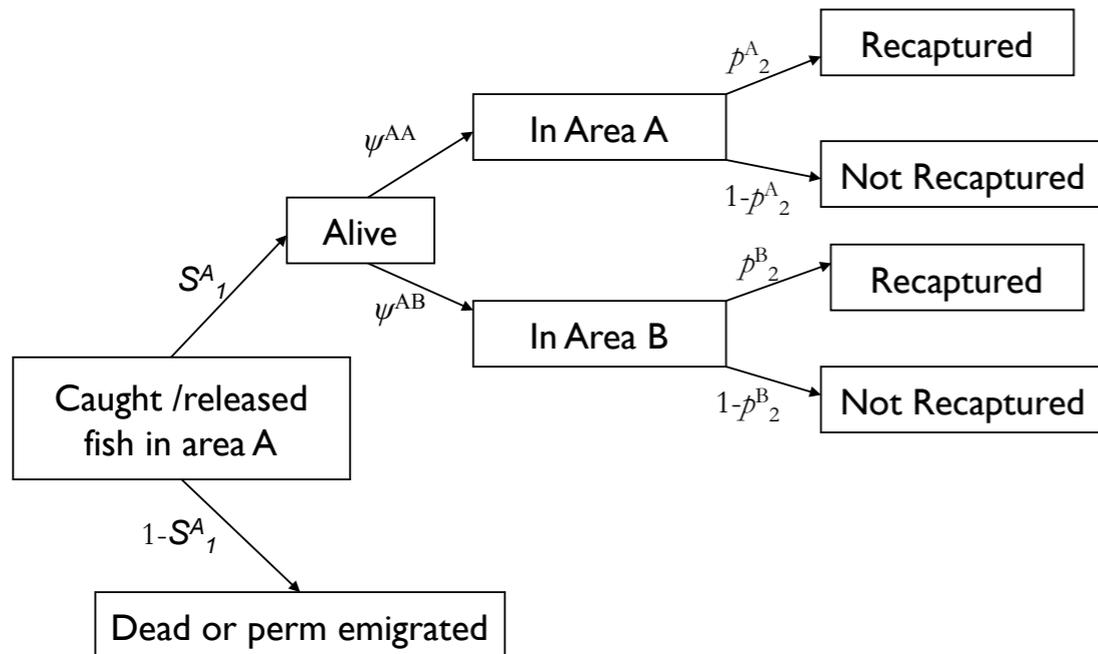
Apparent survival,  $S$  (area)

Movement (transition),  $\psi$

**A multi-state model**



# Probability diagrams



Letters are used in place of “1” to indicate where the fish was captured

e.g., 2 states represented by A, B

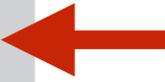
History: A0ABA

Interpretation: initially captured in state (location) ‘A’  
not recaptured second occasion, recaptured 3<sup>rd</sup>  
occasion in state ‘A’, recaptured fourth occasion state  
‘B’, recaptured fifth occasion state ‘C’

# Some basic probabilities of survival analyses

$P(111)$  = Individual was captured on 1st (marked), 2nd and 3rd occasions.

Type of model	Capture history (CH)	Probability of CH
<b>Closed population</b>	111 010	$p_1 * p_2 * p_3$
<b>Open population (CJS)</b>		

- Closed pop model:
- no gains/loss to N
  - Estimate abundance (N) and p
- 

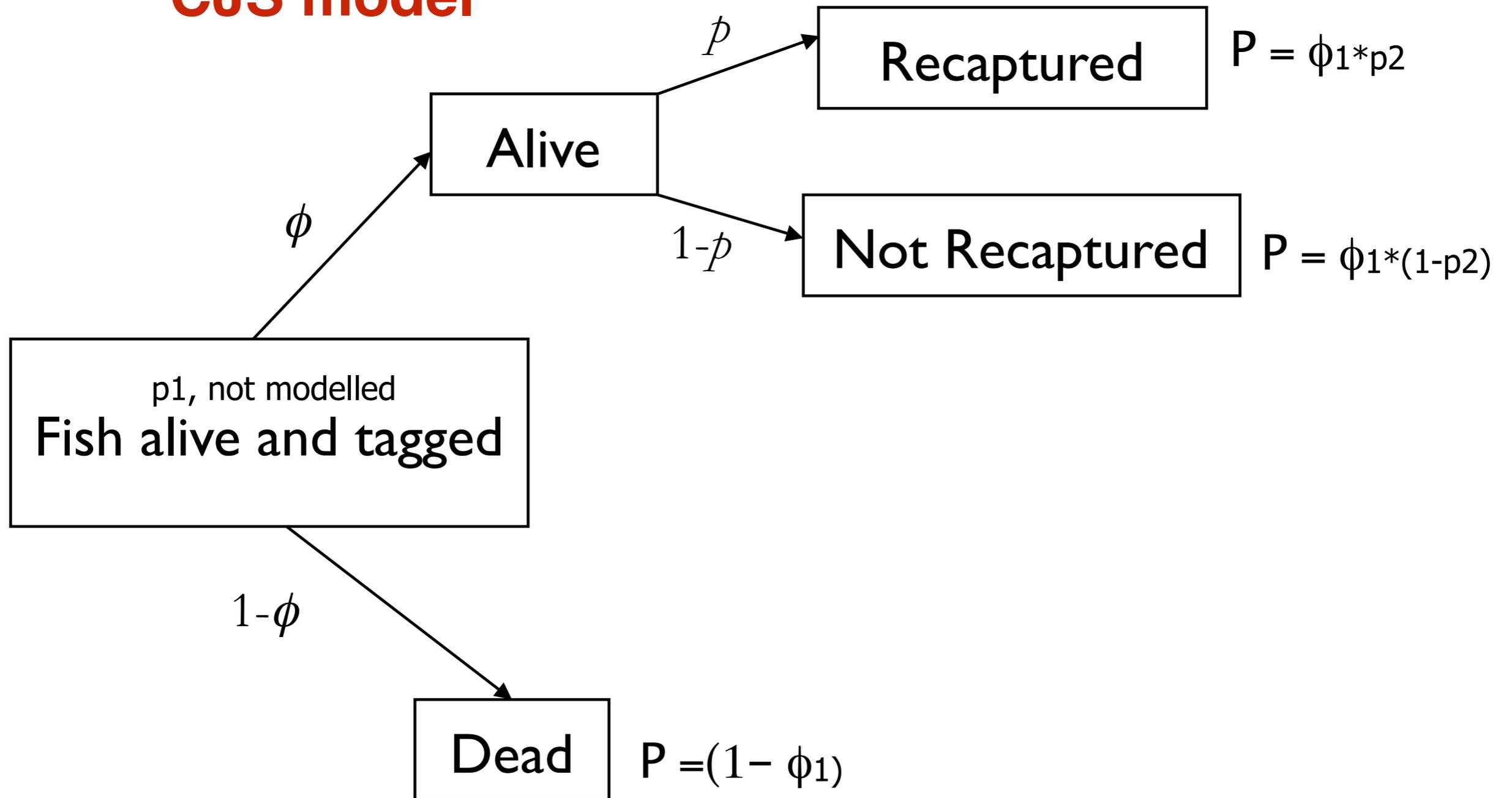
# Some basic probabilities of survival analyses

$P(010)$  = individual was not captured on first occasion, was captured on 2nd, not capt. on 3rd.

Type of model	Capture history (CH)	Probability of CH
<b>Closed population</b>	111 010	$p_1 * p_2 * p_3$ $(1-p_1) * p_2 * (1-p_3)$ ←
<b>Open population (CJS)</b>		

# Probability diagrams

## CJS model



# Some basic probabilities of survival analyses

$P(111)$  = survive 1st interval, detected second interval, survive 2nd and detected in 3rd

Type of model	Capture history (CH)	Probability of CH
Closed population	111	$p_1 * p_2 * p_3$
	101	$p_1 * (1 - p_2) * p_3$
Open population (CJS)	111	$\phi_1 * p_2 * \phi_2 * p_3$
	101	$\phi_1 * (1 - p_2) * \phi_2 * p_3$
	100	$(1 - \phi_1) + \phi_1 * (1 - p_2) * (1 - \phi_2 * p_3)$



- Open pop model:
- Death happens
  - Estimate Survival (S) and p
  - Need to survive interval before detection can occur

# Some basic probabilities of survival analyses

$P(101)$  = survive 1st interval, but not detected second interval, but survive 2nd and detected in 3rd

Type of model	Capture history (CH)	Probability of CH
Closed population	111	$p_1 * p_2 * p_3$
	101	$p_1 * (1 - p_2) * p_3$
Open population (CJS)	111	$\phi_1 * p_2 * \phi_2 * p_3$
	101	$\phi_1 * (1 - p_2) * \phi_2 * p_3$
	100	$(1 - \phi_1) + \phi_1 * (1 - p_2) * (1 - \phi_2 * p_3)$



- Open pop model:
- Death happens
  - Estimate Survival (S) and p
  - Need to survive interval before detection can occur

# Some basic probabilities of survival analyses

$P(100)$  = either die in 1st interval, OR survive 1st but not detected second, then either died 2nd, or lived but not detected 3rd

Type of model	Capture history (CH)	Probability of CH
Closed population	111	$p_1 * p_2 * p_3$
	101	$p_1 * (1 - p_2) * p_3$
Open population (CJS)	111	$\phi_1 * p_2 * \phi_2 * p_3$
	101	$\phi_1 * (1 - p_2) * \phi_2 * p_3$
	100	$(1 - \phi_1) + \phi_1 * (1 - p_2) * (1 - \phi_2 * p_3)$

- Open pop model:
- Death happens
  - Estimate Survival (S) and p
  - Need to survive interval before detection can occur



# Many capture histories...

	Probability of recapture, first, in time period $x$		
Cohort released	2	3	4
$R_1$	$\phi_1 p_2$	$\phi_1(1-p_2)^* \phi_2 p_3$	$\phi_1(1-p_2)^* \phi_2(1-p_3)^* \phi_3 p_4$
$R_2$		$\phi_2 p_3$	$\phi_2(1-p_3)^* \phi_3 p_4$
$R_3$			$\phi_3 p_4$

# Multinomial distribution

Capture history  $X$



$$P(X = \{x_1, x_2, \dots, x_c\}) = \frac{N!}{(x_1!)(x_2!)\dots(x_c!)} p_1^{x_1} p_2^{x_2} \dots p_c^{x_c}$$

# Multinomial distribution

Probability of each capture history



$$P(X = \{x_1, x_2, \dots, x_c\}) = \frac{N!}{(x_1!)(x_2!)\dots(x_c!)} p_1^{x_1} p_2^{x_2} \dots p_c^{x_c}$$

The MLE estimator



$$\Pr(\{x_\omega\} | R_1) = \frac{R_1!}{\prod_\omega x_\omega!} (\phi_1 p_2 \phi_2 p_3)^{x_{111}} [\phi_1 p_2 (1 - \phi_2 p_3)]^{x_{110}} \\ \times [\phi_1 (1 - p_2) \phi_2 p_3]^{x_{101}} (x_1)^{x_{100}}$$

Coding challenge